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CONCERNING GREEN'S THEOREM AND THE CAUCHY-RIEMANN DIFFERENTIAL EQUATIONS

BY M. B. PORTER

CONSIDER a region R , bounded by a continuous closed curve C , and lying between the lines $y = a$ and $y = b$. If the curve C is such that lines parallel to the x -axis cut it in only a finite number of points we shall show that :

If $f(xy)$ and $f'_x(xy)$ be defined for all those points within R lying on the lines $y = y_k$, where the points y_k are everywhere dense in (ab) , and if, $f(xy)$ being defined for all points of C , the integrals $\int_C f(xy) dy^*$ and $\int_R \int f'_x(xy) dx dy$ exist in Riemann's sense, then

$$\int_C f(xy) dy = \int_R \int f'_x(xy) dx dy. \quad (1)$$

It will suffice to give the proof in the case $y = y_k$ cuts C in at most two points x_i, x_i .

We have by definition,

$$\int_C f(xy) dy = \lim \Sigma \left[f(x_i y_k) - f(x_{i-1} y_k) \right] \Delta y_k, \quad x_i \geq x_{i-1}.$$

But

$$f(x_i y_k) - f(x_{i-1} y_k) = f(x_i y_k) - f(x_{i+1} y_k) + f(x_{i+2} y_k) - f(x_{i+1} y_k) + \dots + f(x_{i-1} y_k) - f(x_i y_k)$$

identically, which on applying the Mean Value Theorem becomes :

$$f'_x(x'_i y_k) \Delta x_i + f'_x(x'_{i+1} y_k) \Delta x_{i+1} + \dots + f'_x(x'_{i-1} y_k) \Delta x_i,$$

where

$$x_s < x'_s < x_{s+1} \quad \text{and} \quad \Delta x_s = x_{s+1} - x_s,$$

so that

$$\int_C f(xy) dy = \lim \Sigma_R \Sigma f'_x(x'_s y_k) \Delta x_s \Delta y_k. \quad (2)$$

*For a curve of type C , or more generally for any curve whose ordinate $y = \phi(t)$ is a continuous function of *limited variation* of t , a sufficient condition for the existence of this integral is that $f(xy)$ be continuous on C . Cf. Vallée-Poussin: *Cours d'Analyse*, I, p. 313.

The right-hand side is, by hypothesis, equal to $\int_R \int f'_x(xy) dx dy$, which proves equation (1).

In case $y = y_k$ cuts C in $2m$ points we should arrive at (2) by applying the identity and Mean Value Theorem to each of the m segments of $y = y_k$ lying inside of C .

We have hitherto supposed that f'_x was limited in R . If we suppose however that $|f'_x(xy)| < K$ on the lines $y = y_k$ but becomes infinite for point sets lying on lines $y = \bar{y}_k$ of the set *complementary* to $[y_k]$ in (ab) , then (1) will still hold if $f'_x(xy)$ be *improperly* integrable over R .

The proof is simple: Since the point set over which f'_x becomes infinite must be of content zero, if we write (2) in the form

$$\int_C f(xy) dy = \lim \Sigma \Sigma' f'_x(x'_s y_k) \Delta x_s \Delta y_k + \lim \Sigma \Sigma'' f'_x(x'_s y_k) \Delta x_s \Delta y_k,$$

where the second double sum applies to those rectangles in which f'_x becomes infinite and the first applies to the remaining rectangles, we see that the second limit will be zero since $|f'_x(x'_s y_k)| < K$ while the first will be the improper integral $\int_R \int f'_x dx dy$.

In conclusion we note that if the theory of analytic functions be founded on the Cauchy-Riemann differential equations

$$\left. \begin{aligned} u'_x &= v'_y \\ u'_y &= -v'_x \end{aligned} \right\}, \quad (3)$$

to establish the fundamental relation

$$\int_C (u + iv)(dx + idy) = 0 \quad \left[u + iv = f(z) \right],$$

it will suffice to postulate that u and v are single valued and continuous and that the derivatives in (3) are limited, integrable, and defined over sets of parallel lines $x = [x_k]$, $y = [y_k]$ which are everywhere dense in the region. The existence of $f'(z)$ need not be postulated for any point of the region.